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Diffusion of a chemically reactive species from a stretching sheet

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Abstract—The transfer of chemically reactive species in the laminar flow over an elastic plane surface is considered. The viscous flow is driven solely by the linearly stretched surface and the reactive species is emitted from this sheet and undergoes an isothermal and homogeneous one-stage reaction as it diffuses into the surrounding fluid. A similarity transformation is devised, which reduces the concentration conservation equation to an ordinary differential equation. An exact analytical solution due to Crane [*Z. Angew. Math. Phys.* **21**, 645–647 (1970)] is adopted for the velocity, whereas the concentration field is obtained numerically. The computations showed that the principal effect of a destructive chemical reaction is to reduce the thickness of the concentration boundary layer and to increase the mass transfer rate from the stretching sheet to the surrounding fluid. This effect appeared to be more pronounced for a first-order reaction than for second- and third-order reactions. A nonuniqueness of the concentration distributions for generative first-order reactions was revealed by the computations.

1. INTRODUCTION

THE TRANSPORT of heat, mass and momentum in laminar boundary layers on moving inextensible or stretching surfaces has considerable practical relevance in, for example, electrochemistry [1, 2] and polymer processing [3, 4]. The majority of the studies of these transport processes have so far been devoted to flows induced by surfaces moving with a constant velocity. Crane [5], however, considered the laminar boundary layer flow of a Newtonian fluid caused by a flat, elastic sheet whose velocity varies linearly with the distance from a fixed point on the sheet. The same problem has more recently been extended to fluids obeying non-Newtonian constitutive equations; i.e. Rivlin–Ericksen fluids [6], micropolar fluids [7], second-order fluids [8], Walters' liquid B′ [9], and power-law fluids [10].

The heat transfer problem associated with the Newtonian boundary layer flow past a stretching sheet has been studied by several authors, e.g. refs. [5, 11–16]. By taking advantage of the mathematical equivalence of the thermal boundary layer problem with the concentration analogue, results obtained for heat transfer characteristics can be carried directly over to the case of mass transfer by replacing the Prandtl number by the Schmidt number. However, the presence of a chemical reaction term in the mass diffusion equation generally destroys the formal equivalence with the thermal energy problem and, moreover, generally prohibits the construction of the otherwise attractive similarity solutions. Chambré and Young [17], for example, considered diffusion of a reactive species into the fluid flow past a wedge-shaped body and concluded that a similarity solution exists only in the case of stagnation point flow treated earlier by Chambré [18]. Since numerical techniques for nonsimilar problems are significantly more time consuming than solution schemes devised for sets of ordinary differential equations, Dural and Hines [19] recently advocated approximate methods like the Method of Weighted Residuals for many practical purposes.

In the present study the transfer of a chemically reactive species in the laminar flow over a linearly stretching surface is considered. The reactive component given off by the surface undergoes an isothermal and homogeneous one-stage reaction as it diffuses into the surrounding fluid. By taking advantage of an explicit analytical solution of the momentum boundary layer problem [5], it will be demonstrated that similarity can be achieved also for the concentration field. Accurate numerical solutions of the resulting non-linear ordinary differential equation will be provided for first- and higher-order reactions at various rates, covering the range of Schmidt numbers from 0.01 to 100. Similarity solutions are generally rare and, to the best of our knowledge, no such solution has so far been published for the concentration distribution in the presence of higher-order reactions.

2. MATHEMATICAL FORMULATION

We consider the steady and incompressible flow of a Newtonian fluid past a flat and impermeable elastic

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NOMENCLATURE

a	constant [s^{-1}]
c	concentration [$\mathrm{kg\,m}^{-3}$]
D	molecular diffusion coefficient [$\mathrm{m}^2\,\mathrm{s}^{-1}$]
f	dimensionless stream function
k_n	reaction-rate constant [$\mathrm{s}^{-1}(\mathrm{kg\,m}^{-3})^{1-n}$]
n	order of reaction
Nu_x	local Nusselt number, equation (15)
Re_x	local Reynolds number, ax^2/ν
Sc	Schmidt number, ν/D
u	velocity component along the sheet
v	velocity component normal to the sheet
x	coordinate along the sheet
y	coordinate normal to the sheet.

Greek symbols

β	dimensionless reaction-rate parameter, $k_n c_w^{n-1}/a$
η	similarity variable, equation (6b)
Θ	dimensionless concentration, c/c_w
ν	kinematic fluid viscosity [$\mathrm{m}^2\,\mathrm{s}^{-1}$]
ψ	stream function [$\mathrm{m}^2\,\mathrm{s}^{-1}$].

Subscripts

c	concentration layer
w	wall condition
x	local value.

sheet. By applying two equal and opposite forces along the x-axis, the sheet is being stretched with a speed proportional to the distance from the fixed origin $x = 0$. The resulting motion of the otherwise quiescent fluid is thus caused solely by the moving surface. The continuity and momentum equations governing the flow in the viscous boundary layer along the stretching sheet become:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

subject to the boundary conditions

$$u(x,0)=ax \tag{3a}$$

$$v(x,0)=0 \tag{3b}$$

$$u(x,y)\to 0 \quad \text{as} \quad y\to\infty. \tag{3c}$$

Here, u and v are the components of the fluid velocity in the x- and y-directions, respectively, whereas ν denotes the constant kinematic viscosity of the fluid and a is a positive constant. It has implicitly been assumed that the diffusion rate at the stretching sheet results in a negligible normal velocity component.

The concentration field $c(x, y)$ is governed by the diffusion equation, which in the boundary layer approximation reduces to:

$$u\frac{\partial c}{\partial x}+v\frac{\partial c}{\partial y}=D\frac{\partial^2 c}{\partial y^2}-k_n c^n \tag{4}$$

where D is the diffusion coefficient of the diffusing species in the fluid, and k_n denotes the reaction-rate constant of an nth-order homogeneous and irreversible reaction. Since the concentration of the reactant is maintained at a prescribed constant value c_w at the sheet and is assumed to vanish far away, the relevant boundary conditions for the concentration equation (4) become:

$$c(x,0)=c_w \tag{5a}$$

$$c(x,y)\to 0 \quad \text{as} \quad y\to\infty. \tag{5b}$$

In accordance with the classical boundary layer arguments, streamwise diffusion of momentum and mass concentration has been neglected in the respective conservation equations (2) and (4).

3. SOLUTION OF THE MOMENTUM BOUNDARY LAYER PROBLEM

Let us now introduce new similarity variables f and η such that

$$\psi=(a\nu)^{1/2}xf(\eta) \tag{6a}$$

$$\eta=(a/\nu)^{1/2}y. \tag{6b}$$

The velocity components u and v can then be related to the physical stream function ψ according to its definition:

$$u=\partial\psi/\partial y=axf'(\eta) \tag{7a}$$

$$v=-\partial\psi/\partial x=-(a\nu)^{1/2}f(\eta). \tag{7b}$$

The continuity equation (1) is thereby automatically satisfied, whereas the momentum equation (2) transforms into the ordinary differential equation:

$$f'''-(f')^2+ff''=0 \tag{8}$$

subject to the boundary conditions

$$f(0)=0 \tag{9a}$$

$$f'(0)=1 \tag{9b}$$

$$f'(\eta)\to 0 \quad \text{as} \quad \eta\to\infty \tag{9c}$$

where the prime signifies differentiation with respect to η.

The above momentum boundary layer problem, which is uncoupled from the mass transfer problem, exhibits the simple similarity solution

$$f(\eta)=1-\exp(-\eta) \tag{10}$$

obtained by Crane [5]. Here, it may be worthwhile to mention that the exact analytical solution of equations (8) and (9) was first noticed by Stuart [20] in a quite different context, while the uniqueness of this elegant solution was proved more recently by McLeod and Rajagopal [21] and Troy *et al.* [22].

4. SOLUTION OF THE CONCENTRATION BOUNDARY LAYER PROBLEM

Assuming that similarity can be achieved also for the mass concentration problem, we introduce the transformation:

$$c = c_w \Theta(\eta) \qquad (11)$$

where $\Theta(\eta)$ is a dimensionless concentration field. The non-linear partial differential equation (4) can now be represented by the ordinary differential equation:

$$\Theta'' + Scf\Theta' = \beta Sc\Theta^n \qquad (12)$$

with boundary conditions

$$\Theta(0) = 1 \qquad (13a)$$

$$\Theta(\eta) \to 0 \quad \text{as} \quad \eta \to \infty. \qquad (13b)$$

Here, $Sc = \nu/D$ is the Schmidt number and $\beta = k_n c_w^{n-1}/a$ is a reaction-rate parameter.

Evidently, the concentration field Θ is coupled to the velocity field through the dimensionless stream function f in the non-linear mass transfer equation (12). However, in the special case of a non-reacting species ($\beta = 0$) the non-linear term on the right-hand side of equation (12) vanishes, and the present concentration boundary layer problem becomes formally equivalent with the analogous thermal boundary layer problem. The latter problem, in which the Prandtl number replaces the Schmidt number, has been considered by several authors [5, 11–16]. The analytical solution

$$\Theta = [1 - \exp(-e^{-\eta})]\, e/(e-1) \qquad (14)$$

for $Sc = 1$ was apparently first obtained by Crane [5]. Numerical results for the wall gradient $\Theta'(0)$ for some other Schmidt numbers were provided independently by Vleggaar [11] and Gupta and Gupta [12] and more recently by Jeng *et al.* [15], while series solutions for $\Theta(\eta)$ in terms of Kummer's functions and incomplete gamma functions were developed by Grubka and Bobba [14] and Chen and Char [16], respectively.

In the general case $\beta \neq 0$, the presence of the non-linearity in equation (12) makes numerical integrations inevitable. However, with the solution of the momentum boundary layer problem already given by equation (10), the concentration equation (12) is written as a system of two first-order equations and solved numerically by means of a standard fourth-order Runge–Kutta integration technique.

5. RESULTS AND DISCUSSIONS

The non-linear concentration equation (12) subject to the boundary conditions (13) has been solved numerically for five different values of the Schmidt number in the range $0.01 \leqslant Sc \leqslant 100$ and for five values of the reaction-rate parameter $\beta \leqslant 10$, for first-, second- and third-order reactions. The effects of the three independent parameters Sc, β and n on the concentration distribution are shown in Figs. 1–3,

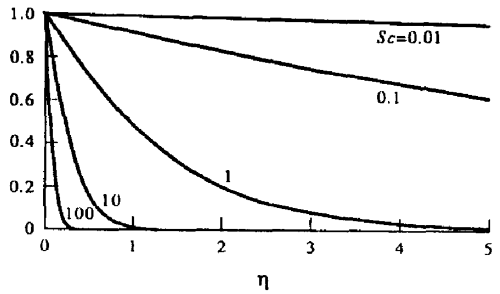

FIG. 1. Concentration profiles $\Theta(\eta)$ for a non-reactive species ($\beta = 0$) with the Schmidt number as parameter.

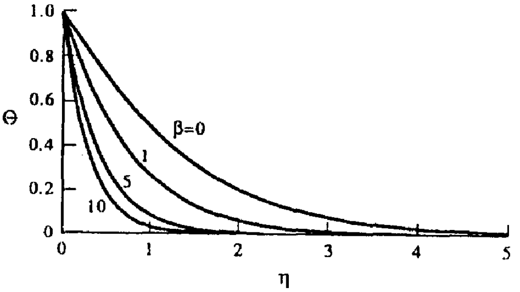

FIG. 2. Concentration profiles $\Theta(\eta)$ for destructive first-order reactions ($n = 1$) for $Sc = 1$ with the reaction-rate β as parameter.

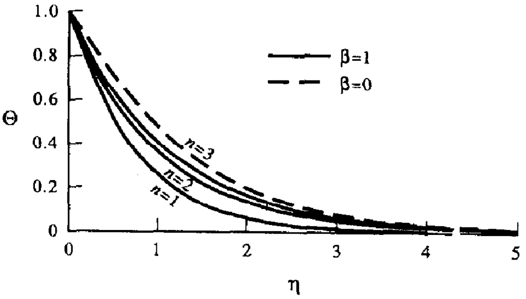

FIG. 3. Concentration profiles $\Theta(\eta)$ for destructive reactions of different orders n with $\beta = 1$ and $Sc = 1$. The broken line represents the non-reactive situation for $Sc = 1$.

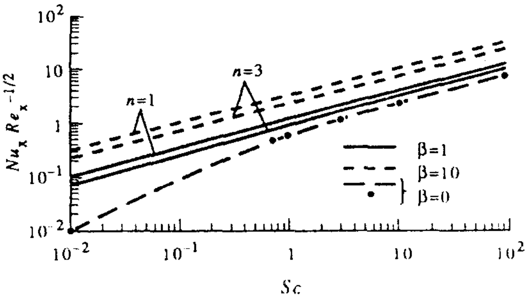

FIG. 4. Dependence of local Nusselt number on Schmidt number. The symbols denote numerical results of Grubka and Bobba [14] for the non-reactive case $\beta = 0$.

respectively, whereas Fig. 4 summarizes the computed wall-gradients $-\Theta'(0)$ reported in Table 1. Since the local mass transfer from the surface to the fluid is conventionally expressed in dimensionless form as a local Nusselt number:

$$Nu_x \equiv -\frac{x}{c_w}\cdot\frac{\partial c}{\partial y}\bigg|_{y=0} = -\Theta'(0)\cdot Re_x^{1/2} \qquad (15)$$

where

$$Re_x \equiv ax^2/\nu \qquad (16)$$

is a local Reynolds number based on the surface velocity ax, the wall-gradient $\Theta'(0)$ becomes a crucial quantity in mass transfer analysis. For the sake of completeness, the dimensionless thickness η_c of the concentration boundary layer is reported in Table 2 for the same range of parameter values as those in Table 1. Here, η_c is defined as the value of the similarity variable η at which the dimensionless concentration Θ has been reduced to 0.01.

The concentration profiles in the non-reacting case ($\beta = 0$) in Fig. 1 show the expected Schmidt number dependence, i.e. the thickness of the concentration boundary layer decreases with Sc, whereas the magnitude of the wall-gradient increases. This is in agreement with the equivalent heat transfer problem, for which numerical data of Grubka and Bobba [14] have been plotted in Fig. 4.

In the limiting case $Sc \to 0$ the concentration boundary layer becomes so thick compared to the momentum layer that the decay of the concentration from its sheet value c_w takes place in fluid at uniform velocity $v = -(a\nu)^{1/2}$ directed perpendicular to the stretching sheet. Thus, with $f = 1$ in equation (12), the asymptotic solution

$$Nu_x = Re_x^{1/2}\cdot Sc \qquad (17a)$$

can readily be derived.

On the contrary, in the low diffusivity or high Schmidt number limit, the concentration gradients are contained within the innermost part of the velocity boundary layer, in which $f = 1-\exp(-\eta) \approx \eta$. The local Nusselt number therefore tends to

$$Nu_x = \sqrt{\left(\frac{2}{\pi}\right)}Re_x^{1/2}\,Sc^{1/2} \qquad (17b)$$

in the limit as $Sc \to \infty$. The latter asymptote is consistent with the leading term in a series expansion for high Prandtl numbers for the equivalent heat transfer problem [13]. Interestingly, the asymptotic formulas for the extreme Schmidt number regimes exhibit a different Sc dependence than those for boundary layers over stationary surfaces.

The effect of a *destructive* chemical reaction, i.e. $\beta > 0$, is to reduce the thickness of the concentration layer and increase the wall transfer, as shown for a first-order reaction in Fig. 2. However, it can be observed from Fig. 3 that the effect of a first-order

Table 1. Computed values of $Nu_x\ Re_x^{-1/2}$

β	0	0.01			0.1			1.0			10		
Sc		$n = 1$	$n = 2$	$n = 3$	$n = 1$	$n = 2$	$n = 3$	$n = 1$	$n = 2$	$n = 3$	$n = 1$	$n = 2$	$n = 3$
0.01	0.0099	0.0160	0.0138	0.0127	0.0367	0.0301	0.0261	0.1043	0.0851	0.0736	0.3193	0.2607	0.2257
0.10	0.0913	0.0998	0.0959	0.0944	0.149	0.129	0.118	0.348	0.286	0.249	1.017	0.831	0.720
1.0	0.582	0.592	0.588	0.587	0.669	0.636	0.622	1.177	1.000	0.907	3.232	2.649	2.303
10	2.308	2.334	2.327	2.324	2.509	2.440	2.409	3.880	3.400	3.157	10.25	8.418	7.352
100	7.766	7.886	7.870	7.861	8.395	8.202	8.112	12.51	11.07	10.35	32.47	26.70	23.36

Table 2. Computed thicknesses η_c of the concentration boundary layer

β	0	0.01			0.1			1.0			10		
Sc		$n = 1$	$n = 2$	$n = 3$	$n = 1$	$n = 2$	$n = 3$	$n = 1$	$n = 2$	$n = 3$	$n = 1$	$n = 2$	$n = 3$
0.01	461	285	424	447	124	317	399	43.8	167	311	14.3	63.9	207
0.10	46.1	42.3	45.7	46.0	28.5	42.5	44.8	12.5	31.8	39.9	4.41	16.7	31.2
1.0	5.06	5.01	5.05	5.06	4.64	5.00	5.04	3.12	4.61	4.87	1.34	3.41	4.29
10	0.978	0.975	0.977	0.977	0.955	0.973	0.976	0.804	0.940	0.960	0.412	0.797	0.892
100	0.272	0.271	0.272	0.272	0.267	0.271	0.271	0.235	0.264	0.268	0.129	0.230	0.252

reaction is more pronounced than of higher-order reactions ($n = 2$ and 3). This observation is in keeping with the approximate solutions for a reactive flat-plate boundary layer due to Dural and Hines [19].

For large values of the reaction rate parameter β, equation (12) becomes essentially a balance between diffusion and reaction, while the flow dependent convection contributes marginally to the concentration budget. Thus, if the convective transport term $Scf\Theta'$ is neglected, equation (12) can be integrated for $n = 1$ to yield:

$$\Theta = \exp[-(\beta Sc)^{1/2}\eta] \tag{18a}$$

with the associated surface flux

$$Nu_x = Re_x^{1/2}(\beta Sc)^{1/2} \tag{18b}$$

and the thickness of the concentration layer being

$$\eta_c = -(\beta Sc)^{-1/2}\ln(0.01) \tag{18c}$$

for first-order reactions.

The accuracy of the approximate formulas (18) can be examined by comparisons with the numerical values provided in Table 1 and Table 2 obtained by direct numerical integration of the complete concentration budget (12). For $\beta = 10$ the Nusselt number is slightly underpredicted by equation (18b) while the thickness of the concentration layer is somewhat overestimated by equation (18c). The greatest errors, -2.6% and $+12.9\%$, respectively, occurred for $Sc = 100$. Although the nearly straight solid lines in Fig. 4 may suggest that (18b) is applicable also for $\beta = 1$, the accuracy of the approximate formula deteriorates with decreasing β. For $\beta = 1$, for example, the Nusselt number is underestimated by 4 and 20% for $Sc = 0.01$ and 100, respectively.

It is noteworthy that an exact analytical solution of the complete concentration equation (12) exists for the particular case $Sc = 1$, $\beta = 1$ and $n = 2$, namely:

$$\Theta = \exp(-\eta). \tag{19}$$

Incidentally, the concentration distribution becomes identical to the profile of the streamwise velocity component f' in this case. The exact solution (19) of the non-linear ordinary differential equation (12) is useful as a reference against which approximate solutions and numerical computations may be checked. Moreover, it is readily shown from equation (19) that the diffusion Θ'' of the reactive species from the sheet and into the fluid decays as $\exp(-\eta)$, whereas the destruction $\beta Sc\Theta^n$ of the species by the second-order reaction decreases as $\exp(-2\eta)$ with the distance from the sheet. The excess amount of the reactant diffused from the surface and not destroyed by the chemical reaction is exactly balanced by the convective transport of dilute fluid towards the sheet. Unlike the two other terms in the concentration budget, the convective contribution vanishes at the sheet as well as at infinity. Its greatest contribution, which occurs as $\eta = \ln 2 \cong 0.6931$, happens to equal the amount of reactant being destroyed in this position.

The governing concentration equation (12) is valid also for *generative* reactions ($\beta < 0$), i.e. the species which diffuses from the stretching sheet is also produced by the chemical reaction in the stream. Here, we consider a reaction of first order ($n = 1$), for which the reactive species has Schmidt number $Sc = 1.0$. Some computed profiles for $\beta = -0.10$ are displayed in Fig. 5. However, an infinite number of solutions are available, all of which satisfy both boundary conditions defined in equation (13). Three qualitatively different concentration profiles were obtained, depending on the choice of the wall-gradient $\Theta'(0)$. First, for moderate negative wall-gradients monotonically decaying distributions were observed, whereas profiles exhibiting a distinct maximum were found for positive wall-gradients. In the latter case, the reaction is so efficient that the highest concentration occurs in the stream rather than at the surface. The species is accordingly diffused from the concentration peak and towards the sheet. Finally, highly negative wall-gradients may lead to a definite minimum in the distribution of the concentration. These solutions are, however, not physically realistic since they exhibit regions with negative Θ.

The concentration distributions presented in Fig. 6 for $\beta = -1.0$ differ from those in Fig. 5 in having an oscillating behaviour for large values of η. As for

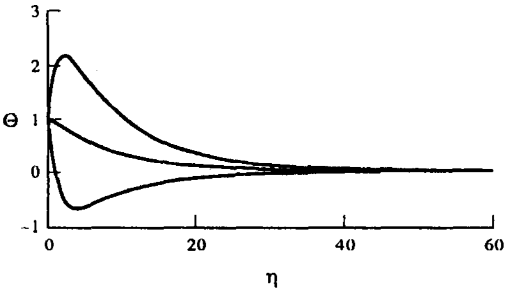

Fig. 5. Representative concentration profiles $\Theta(\eta)$ for generative first-order reactions with $\beta = -0.1$ and $Sc = 1.0$.

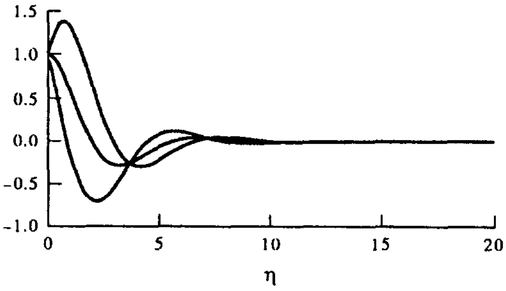

Fig. 6. Representative concentration profiles $\Theta(\eta)$ for generative first-order reactions with $\beta = -1.0$ and $Sc = 1.0$.

$\beta = -0.10$, an infinite number of solutions exist, each being determined by the wall-gradient. For the higher generative reaction-rate ($\beta = -1.0$) the oscillatory decay to zero makes the concentration negative in localized areas and these solutions are accordingly not meaningful from a physical point of view.

6. CONCLUDING REMARKS

The concentration boundary layer along a linearly stretching sheet is ideally suited for the assessment of approximate mathematical or numerical approaches to reactive boundary layers since the velocity distribution is given exactly in analytical form. The nonlinear mass concentration equation transforms into an ordinary differential equation, which can be solved numerically to an arbitrary degree of accuracy. The computed results show that the principal effects of a destructive chemical reaction are to reduce the thickness of the concentration boundary layer and increase the mass transfer rate from the stretching sheet to the surrounding fluid. It can also be concluded that these effects are more pronounced for a first-order reaction than for higher-order reactions. The nonuniqueness of the concentration distribution for generative first-order reactions has been demonstrated. An extension of the present analysis to viscoelastic fluids is published in an accompanying paper [23].

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